Fixed vs. Random Factors

Fixed:
- All levels of interest are included in the experiment.
- Interest is in *means*.
- Treatment effects sum to 0.
- Inference is over levels in the experiment.

Random:
- Levels are randomly selected from larger population.
- Interest is in *variances*.
- Treatment effects are normal (0, σ^2).
- Inference is over levels in population.

Three Model Types

Fixed – all factors and interactions are fixed effects (except residual error)

Random – all factors and interactions are random (no fixed factors in model)

Mixed – contains at least one fixed factor and two or more random factors (including residual) $\therefore$ contains mixed interactions between fixed and random factors

Expected Mean Squares Algorithm
Setting up the Table

1. Write the linear model for the experiment.
2. Create a table with *c* columns where *c* = 2 + the # subscripts and *r* rows where *r* = number of terms in model.
3. Label the column on left **Source**, the one to the far right, **EMS**. The remaining columns in between are labeled with subscripts used in the model (eg. ijk).
4. In the column heading above each subscript use an F or R to indicate whether the factor associated with it is fixed or random.
5. Above these, write the number of levels associated with the subscript.
6. Enter sources of variation from the model in left-hand column, excluding the mean and including subscripts for each term.
7. In the subscript columns:
    a. Enter a 1 in each cell where the column intersects a row where the subscript occurs in parentheses.
    b. Enter a 1 in each cell where a column labeled as **random** intersects a row where the subscript associated with the column occurs in the term.
    c. Enter a 0 in each cell where a column labeled as **fixed** intersects a row where the subscript associated with the column occurs in the term.
    d. Enter the number of treatment levels associated with the column in all remaining cells.

Expected Mean Squares Algorithm
Working the Algorithm

1. Beginning with the last row in the table, construct the expected mean square for each factor listed in the table by:
    a. Only consider including variance components for model terms with all of the subscripts for the factor in question.
    b. To compute the coefficient for a term, multiply the coefficients in the cells to the right of the term **excluding** the cells associated with subscripts included in the term.
    c. If the product is zero, then that term is not included in the variance components for the factor under consideration.
    d. If the product is greater than zero, then that term is included as a variance component in the expected mean square for the factor under consideration.
    e. Write the EMS as a linear combination of variance components in right-hand column using σ to denote random effects and ϕ to denote fixed effects.
2. Continue the steps above for each model term in the left column always working from the bottom upwards.

Expected Mean Squares
Algorithm

Model: $Y_{ijk} = \mu + A_i + B_j + AB_{ij} + \varepsilon_{(ij)k}$

Factors:

Factor A – Fixed

Factor B – Fixed

Reps - Random

Expected Mean Squares
Algorithm

Create table and enter sources of variation from the model.

Source	i	j	k	EMS
A_i				
B_j				
AB_{ij}				
$\varepsilon_{(ij)k}$				

Expected Mean Squares
Algorithm

Enter the number of levels and indicate if fixed or random effect above subscripts.

Source	a F i	b F j	r R k	EMS
A_i				
B_j				
AB_{ij}				
$\varepsilon_{(ij)k}$				

Expected Mean Squares
Algorithm

Enter a 1 in each cell where subscript is contained in parentheses in model term.

Source	a F i	b F j	r R k	EMS
A_i				
B_j				
AB_{ij}				
$\varepsilon_{(ij)k}$	1	1		

Expected Mean Squares
Algorithm

Enter a 0 in cells where the subscripts correspond to a fixed term.

Source	a F i	b F j	r R k	EMS
A_i	0			
B_j		0		
AB_{ij}	0	0		
$\varepsilon_{(ij)k}$	1	1		

Expected Mean Squares
Algorithm

Enter a 1 in cells where the subscripts correspond to a random term.

Source	a F i	b F j	r R k	EMS
A_i	0			
B_j		0		
AB_{ij}	0	0		
$\varepsilon_{(ij)k}$	1	1	1	

Expected Mean Squares
Algorithm

Fill remaining cells with number of levels corresponding to subscript.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	
AB_{ij}	0	0	r	
$\varepsilon_{(ij)k}$	1	1	1	

Expected Mean Squares
Algorithm

Enter σ^2 in the bottom cell on the right. This is the error variance.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	
AB_{ij}	0	0	r	
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Cover the columns corresponding to the AB subscripts. Consider only terms with ij

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	
AB_{ij}	0	0	r	
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Add terms for which the product of uncovered cells > 0. Use ϕ for fixed.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	
AB_{ij}	0	0	r	σ^2
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Add terms for which the product of uncovered cells > 0. Use ϕ for fixed.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	
AB_{ij}	0	0	r	$\sigma^2 + r\phi(AB)$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Repeat for B.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	σ^2
AB_{ij}	0	0	r	$\sigma^2 + r\phi(AB)$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Repeat for B.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	
B_j	a	0	r	$\sigma^2 + ar\phi(B)$
AB_{ij}	0	0	r	$\sigma^2 + r\phi(AB)$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Repeat for A.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	σ^2
B_j	a	0	r	$\sigma^2 + ar\phi(B)$
AB_{ij}	0	0	r	$\sigma^2 + r\phi(AB)$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Repeat for A.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	$\sigma^2 + br\phi(A)$
B_j	a	0	r	$\sigma^2 + ar\phi(B)$
AB_{ij}	0	0	r	$\sigma^2 + r\phi(AB)$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Factor A and B Fixed, Reps random.

Source	a F i	b F j	r R k	EMS
A_i	0	b	r	$\sigma^2 + br\phi(A)$
B_j	a	0	r	$\sigma^2 + ar\phi(B)$
AB_{ij}	0	0	r	$\sigma^2 + r\phi(AB)$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Factor A fixed. Factor B and Reps random.

Source	a F i	b R j	r R k	EMS
A_i	0	b	r	
B_j	a	1	r	
AB_{ij}	0	1	r	
$\varepsilon_{(ij)k}$	1	1	1	

Expected Mean Squares
Algorithm

Factor A fixed. Factor B and Reps random.

Source	a F i	b R j	r R k	EMS
A_i	0	b	r	$\sigma^2 + r\sigma^2_{AB} + br\phi(A)$
B_j	a	1	r	$\sigma^2 + ar\sigma^2_B$
AB_{ij}	0	1	r	$\sigma^2 + r\sigma^2_{AB}$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Factor A fixed. Factor B and Reps random.

Source	a F i	b R j	r R k	EMS
A_i	0	b	r	$\sigma^2 + r\sigma^2_{AB} + br\phi(A)$
B_j	a	1	r	$\sigma^2 + ar\sigma^2_B$
AB_{ij}	0	1	r	$\sigma^2 + r\sigma^2_{AB}$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Factors A, B and Reps random.

Source	a R i	b R j	r R k	EMS
A_i	1	b	r	
B_j	a	1	r	
AB_{ij}	1	1	r	
$\varepsilon_{(ij)k}$	1	1	1	

Expected Mean Squares
Algorithm

Factors A, B and Reps random.

Source	a R i	b R j	r R k	EMS
A_i	1	b	r	$\sigma^2 + r\sigma^2_{AB} + br\sigma^2_A$
B_j	a	1	r	$\sigma^2 + r\sigma^2_{AB} + ar\sigma^2_B$
AB_{ij}	1	1	r	$\sigma^2 + r\sigma^2_{AB}$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
Algorithm

Factors A, B and Reps random.

Source	a R i	b R j	r R k	EMS
A_i	1	b	r	$\sigma^2 + r\sigma^2_{AB} + br\sigma^2_A$
B_j	a	1	r	$\sigma^2 + r\sigma^2_{AB} + ar\sigma^2_B$
AB_{ij}	1	1	r	$\sigma^2 + r\sigma^2_{AB}$
$\varepsilon_{(ij)k}$	1	1	1	σ^2

Expected Mean Squares
3-Factor ANOVA

Model:

$Y_{ijkl} = \mu + A_i + B_j + AB_{ij} + C_k + AC_{ik} + BC_{jk} + ABC_{ijk} + R_{(ijk)l}$

Factors:

A 2
B 4
C 3
R 2

Expected Mean Squares
Direct Tests

A,B, and C Fixed Factors

Source	df	EMS
A_i	1	$\sigma^2 + 24\phi(A)$
B_j	3	$\sigma^2 + 12\phi(B)$
AB_{ij}	3	$\sigma^2 + 6\phi(AB)$
C_k	2	$\sigma^2 + 16\phi(C)$
AC_{ik}	2	$\sigma^2 + 8\phi(AC)$
BC_{jk}	6	$\sigma^2 + 4\phi(BC)$
ABC_{ijk}	6	$\sigma^2 + 2\phi(ABC)$
$\varepsilon_{(ijk)l}$	24	σ^2

Expected Mean Squares

Direct Tests

A Fixed, B and C Random Factors

Source	df	EMS
A_i	1	$\sigma^2 + 2\sigma^2_{ABC} + 8\sigma^2_{AC} + 6\sigma^2_{AB} + 24\phi(A)$
B_j	3	$\sigma^2 + 4\sigma^2_{BC} + 12\sigma^2_{B}$
AB_{ij}	3	$\sigma^2 + 2\sigma^2_{ABC} + 6\sigma^2_{AB}$
C_k	2	$\sigma^2 + 4\sigma^2_{BC} + 16\sigma^2_{C}$
AC_{ik}	2	$\sigma^2 + 2\sigma^2_{ABC} + 8\sigma^2_{AC}$
BC_{jk}	6	$\sigma^2 + 4\sigma^2_{BC}$
ABC_{ijk}	6	$\sigma^2 + 2\sigma^2_{ABC}$
$\varepsilon_{(ijk)l}$	24	σ^2

Expected Mean Squares

Direct Tests

A Fixed, B and C Random Factors

Source	df	EMS
A_i	1	$\sigma^2 + 2\sigma^2_{ABC} + 8\sigma^2_{AC} + 6\sigma^2_{AB} + 24\phi(A)$
B_j	3	$\sigma^2 + 4\sigma^2_{BC} + 12\sigma^2_{B}$
AB_{ij}	3	$\sigma^2 + 2\sigma^2_{ABC} + 6\sigma^2_{AB}$
C_k	2	$\sigma^2 + 4\sigma^2_{BC} + 16\sigma^2_{C}$
AC_{ik}	2	$\sigma^2 + 2\sigma^2_{ABC} + 8\sigma^2_{AC}$
BC_{jk}	6	$\sigma^2 + 4\sigma^2_{BC}$
ABC_{ijk}	6	$\sigma^2 + 2\sigma^2_{ABC}$
$\varepsilon_{(ijk)l}$	24	σ^2

Expected Mean Squares

Approximate Tests

A Fixed, B and C Random Factors

	Source	df	EMS
	A_i	1	$\sigma^2 + 2\sigma^2_{ABC} + 8\sigma^2_{AC} + 6\sigma^2_{AB} + 24\phi(A)$
	B_j	3	$\sigma^2 + 4\sigma^2_{BC} + 12\sigma^2_{B}$
+	AB_{ij}	3	$\sigma^2 + 2\sigma^2_{ABC} + 6\sigma^2_{AB}$
	C_k	2	$\sigma^2 + 4\sigma^2_{BC} + 16\sigma^2_{C}$
+	AC_{ik}	2	$\sigma^2 + 2\sigma^2_{ABC} + 8\sigma^2_{AC}$
	BC_{jk}	6	$\sigma^2 + 4\sigma^2_{BC}$
-	ABC_{ijk}	6	$\sigma^2 + 2\sigma^2_{ABC}$
	$\varepsilon_{(ijk)l}$	24	σ^2

Approximate F test: MS(A)/[MS(AB)+MS(AC)-MS(ABC)]

Approximate F Tests

Satterthwaite's Approximation

$$df = \frac{M^2}{\frac{(a_1 MS_1)^2}{df_1} + \frac{(a_2 MS_2)^2}{df_2} + \dots + \frac{(a_k MS_k)^2}{df_k}}$$

where: $$M = \sum_{i=1}^{k} a_i MS_i$$

Example:

$$df = \frac{[MS(AB) + MS(AC) - MS(ABC)]^2}{MS(AB)^2/3 + MS(AC)^2/2 + MS(ABC)^2/6}$$

Expected Mean Squares

Conservative Tests

A Fixed, B and C Random Factors

Source	df	EMS
A_i	1	$\sigma^2 + \sigma^2_{ABC} + 4\sigma^2_{AC} + 3\sigma^2_{AB} + 12\phi(A)$
B_j	3	$\sigma^2 + 2\sigma^2_{BC} + 6\sigma^2_{B}$
AB_{ij}	3	$\sigma^2 + \sigma^2_{ABC} + 3\sigma^2_{AB}$
C_k	2	$\sigma^2 + 2\sigma^2_{BC} + 8\sigma^2_{C}$
AC_{ik}	2	$\sigma^2 + \sigma^2_{ABC} + 4\sigma^2_{AC}$
BC_{jk}	6	$\sigma^2 + 2\sigma^2_{BC}$
ABC_{ijk}	6	$\sigma^2 + \sigma^2_{ABC}$
$\varepsilon_{(ijk)l}$	0	σ^2